ANALYSIS OF BOOK GRAPH AND WEB GRAPH PROPERTIES

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Abstract. This paper seeks to contribute to the current existing graph theory. We do so by analyzing and explaining certain properties and characteristics of two different graphs. One of which is the Web graph, designed and defined by the authors of this paper, while the other is a more well-known and predefined Book graph. These definitions and defined characteristics should serve as a contribution to previous works of graph theory on a practical level.

1. Introduction

This paper serves as a contribution to graph theory, which was first initiated by Swiss mathematician Leonhard Euler in 1736 [2]. Within this work, we analyze the graph theory of a pre-defined Book graph in addition to a newly defined Web graph. Further, we assess and provide proof for differing properties within these graphs. These proofs as well as these analyses are written with the presupposition that the reader knows basic algebra and simple graph theory.

The Web graph began with a discussion of a grid graph. Our team understood that the concept of coordinate graphs and similar structures are well researched, and wanted to attempt some different approach to the idea. The Web graph could serve as a representation of a coordinate system that is more circular or spiral in nature. This concept could be related to the layout of Paris, France [4]. This would be opposed to the grid-like layout of cities like New York City, NY.

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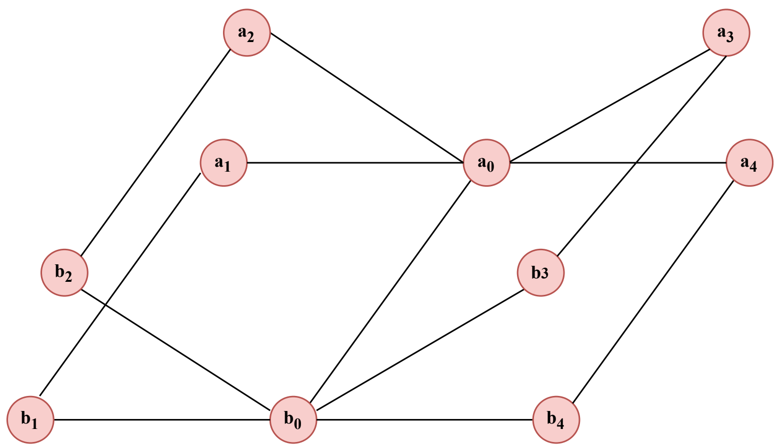
2. Background and definitions

2.1. **Book Graph B_n.** $\forall n \in \mathbb{Z}^*$, a Book Graph B_n is a connected graph with n pages, where one page is a cycle subgraph of four vertices and each page has one common pair of vertices shared among each page subgraph. The common pair of vertices that are shared among each page subgraph will be called the "spine", similar to how the spine of a book connects its pages.

Below is the graph B_4.



2.2. **Definition of B_n.** The vertex set of B_n is defined as:

$$V(B_n) = V_A \cup V_B$$

where

$$V_A = \{a_0, a_1, a_2, ...a_n\}$$

and

$$V_B = \{b_0, b_1, b_2, ...b_n\},$$

each representing a vertex subset making up the 'top' and 'bottom' halves of B_n.

The edge set of B_n is defined as the union of three disjoint edge subsets:

$$A_n = \bigcup_{i=1}^{n} \{a_0, a_i\}, \text{ denoting the 'top' edges,}$$

$$T_n = \bigcup_{i=1}^{n} \{b_0, b_i\}, \text{ denoting the 'bottom' edges,}$$

and

$$P_n = \bigcup_{i=0}^{n} \{a_i, b_i\}, \text{ denoting edges joining the top and bottom vertices,}$$

such that

$$E = A_n \cup T_n \cup P_n.$$

Thus, each vertices of a subset V_A are all connected to a center vertex a_0, and all vertices of a subset V_B are all connected to a center vertex b_0. Finally, each vertex in V_A is connected to a corresponding vertex in V_B.

2.3. **Order and Size of B_n.** Defining the order and size of a book graph, both equations can be properly understood while following along in drawing a graph. When first drawing a book graph, the "spine" of the graph will be drawn. According to the definition of a book graph, the spine contributes an edge, and two vertices as previously defined. This gives a book graph of $n = 0$ an order of 2 and a size of 1.

Further, for each added page, there is a total of two vertices that are added, and three edges, per page. Therefore, the definitions of the order can be designated as:

$$|V(B_n)| = 2n + 2$$

where n is multiplied by 2 to show the two vertices added for each page n, and the trailing added two representing the initial order sourced from the graph's spine. Finally, the size of the graph can be represented by the equation

$$|E(B_n)| = 3n + 1$$

where n is multiplied by 3 to represent the three edges added per page, and the trailing added one represents the one edge in the spine of the graph.

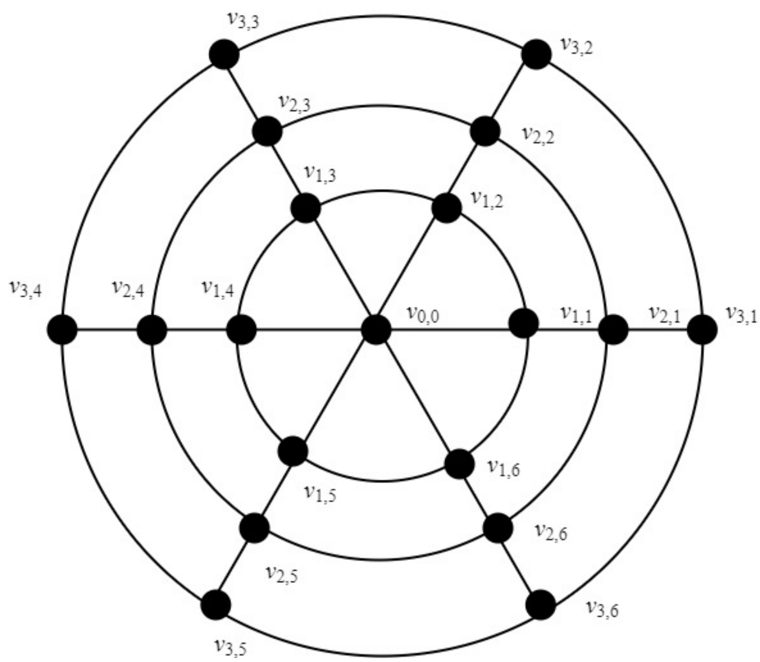
2.4. **Minimum and maximum degree of B_n.** When the book graph has zero pages, i.e. $n = 0$, there are only two vertices: a_0 and b_0, which constitute the spine.

A page in this graph consists of two adjacent vertices where one of them is also adjacent to a_0 and the other is also adjacent to b_0. Therefore, the degree of each page's vertices is 2. Additionally, for each page added to the book graph, the degree a_0 and b_0 will increase by one because they each get one more adjacent vertex. This means that the two vertices on the spine of our book graph will always have a degree of $n+1$ because they are adjacent to each other and on each page. This is our maximum degree.

When $n = 0$, the degree of a_0 and b_0 is one, as they are the only vertices and they are adjacent to each other. Since the degrees of each vertex on a page are always 2 and each page added increases the degree of a_0 and b_0, we know adding additional pages will not give us a vertex with a lower degree than when $n = 0$, where the degree of a_0 and b_0 is 1. Therefore, the minimum degree occurs when $n = 0$, as a value of 1.

2.5. **Web Graph** $W_{n,m}$**.** A web graph $W_{n,m}$ is a connected graph with n rings, and m vertices on each ring, where n must be at least 1 and m cannot be less than 3. Each ring is a cycle sub-graph containing m vertices. If the vertex is not the center vertex, or bullseye vertex, or on the nth ring, each vertex will have a corresponding adjacent inner and outer vertex.

Below is an example for the graph $W_{3,6}$



2.6. **Definition of** $W_{n,m}$**.** The vertex set of $W_{n,m}$ is defined as the union of each ring of vertices, and the center vertex of the graph:

$$V(W_{n,m}) = \bigcup_{i=1}^{n} V_i \cup \{v_0\}$$

where

$$V_i = \{v_1^i, v_2^i, v_3^i, ...v_m^i\}$$

and v_0 is the vertex in the very center or bullseye of the graph.

The edge set can be defined as the union of three distinct subsets of edges: the set of edges around each ring, the set of edges through all the rings, and the set of edges connecting the center, or bullseye vertex to all the vertices on the innermost ring. The formal definition can be defined as:

$$E(W_{n,m}) = \bigcup_{i=1}^{a} E_i \cup \bigcup_{i=1}^{b} F_i \cup H$$

where the edges around the rings, namely *ring-edges*, are:

$$E_i = \{\{v_k^i, v_{k+1}^i\} \mid 1 \leq k \leq m-1\} \cup \{v_1^i, v_m^i\}$$

and the edges connecting each ring, namely *interring-edges*, are:

$$F_i = \{\{v_i^k, v_{k+1}^{k+1}\} \mid 1 \leq k \leq n\}$$

and the edges connecting the center vertex to the innermost ring are (i.e. innermost *interring-edges*):

$$H = \{\{v_0, v_i^1\} \mid 1 \leq i \leq m\}.$$

Also, since each ring is a cycle graph C_m, with m vertices, we can label each ring as $C^1, C^2, C^3, ..., C^n$, from the innermost ring, C^1, to the outermost ring, C^n.

2.7. **Order and Size of $W_{n,m}$.** The *order* of any Web graph $W_{n,m}$ can be calculated as:

$$|V(W_{n,m})| = n \cdot m + 1.$$

This is because, for each of the n rings, there are m vertices that make up a cycle subgraph. The only vertex not part of a ring is the center vertex, which is represented by the addition of 1 to $n \cdot m$.

The *size* of a Web graph evaluates as:

$$|E(W_{n,m})| = 2 \cdot n \cdot m.$$

Counting from the center vertex, there are m edges connecting the center to the innermost ring. Let these edges connecting rings be called *interring edges*. There are exactly m interring edges connecting each inner ring to an outer ring, and there are n rings that require to be connected. So we have

$$\text{total interring edges} = n \cdot m.$$

Counting the number of edges making up each ring, we know that a cycle graph of k vertices always has k edges [3]. Since our rings have m vertices, they have m edges, and we have n rings, therefore,

$$\text{total ring edges} = n \cdot m.$$

Combining these disjoint sets of edges constitutes the set edge with $2 \cdot n \cdot m$ elements.

2.8. **Minimum and Maximum Degree of $W_{n,m}$.** First, there are a few characteristics that should be understood. Firstly, the centermost vertex of the web graph will always have the degree of m. Therefore, due to the bound of $m \geq 3$, the minimum degree of the center vertex is 3. Further, all vertices within the outermost ring, will all have degrees of 3 as well. Two would come from their neighbors among the same ring, and one with its connecting vertex within the next inner ring. Lastly, all vertices in inner rings will thus have a degree of 4. The last degree stems from the edge that leads to a vertex in the next outer ring.

Since neither of the mentioned vertices will have a degree less than 3, this will be designated as the minimum for web graphs. Further, since there is no upper bound on the m variable within a web graph, the degree of the center vertex also has no upper bound. Thus, the maximum of a web graph is infinite.

3. Results

Theorem 3.1. *$\forall$ book graph B_n, where $n \in \mathbb{Z}^*$, the dominating number of is always equal to 2.*

Proof. **Maximal size of dominating set:**
Let B_n be a generic but particular book graph with n positive integer of pages. By definition, the dominating number is the size of the minimal set of vertices needed to connect the whole graph. By definition, B_n has two vertices, a_0 and b_0, constituting the spine of the book. Since all the other vertices are adjacent to either a_0 or b_0, we have

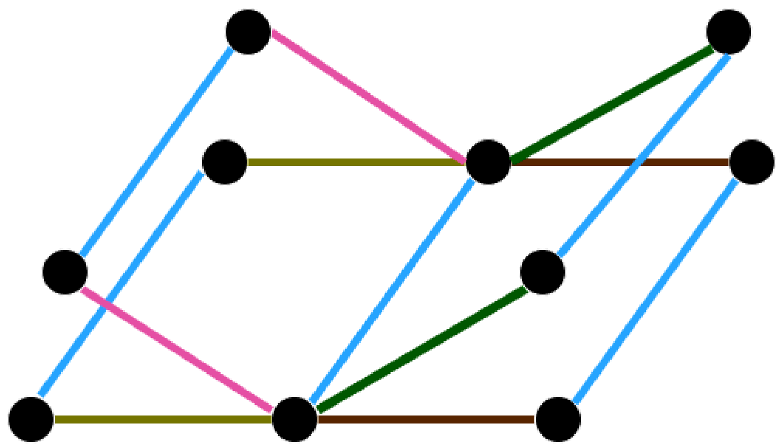
$$\text{Maximal dominating set} = \{a_0, b_0\},$$

providing the upper bound of the dominating number to 2.

Minimal size of dominating set:
Suppose B_n has a dominating number of 1. This implies that there exists vertex v that is adjacent to all the other vertices. Given that the order of B_n is $2n + 2$, v must have a degree of $2n + 1$. However, the maximum degree of a vertex in B_n is $n + 1$. Thus, the minimal size of the dominating set is 2. □

Theorem 3.2. *The chromatic index of a Book graph of size n will be $n+1$*

Proof. Here is an example of how a Book graph can be given a chromatic index of $n+1$. In this particular case, the graph is B_4 and is given a chromatic index of 5.



Suppose not, suppose a Book graph was given a chromatic index of n. By definition of a Book graph, the vertices that make up the spine will have a degree of $n+1$. This indicates there are $n+1$ edges that are incident on the mentioned vertex.

With $n+1$ edges being incident on one vertex, it is impossible to make all of them different colors with a choice of n colors.

Therefore, the chromatic index of a Book graph cannot be n, and thus, is $\geq n+1$.

Further, every edge within the subset A_n of the graph edge set must be a different color since every each within that set is incident on vertex a_0. These same colors can be mirrored by the edges within the subset T_n since they all share b_0, but share no vertices with the edges within A_n. That brings the current total of colors to n.

Lastly, the edges in P_n can be given the same color since they have no shared vertices, as long as that color is different from the already defined n amount. This is due to every vertex contained in P_n is contained also in A_n or T_n. Therefore, since the chromatic index must be $\geq n+1$ and $\leq n+1$, the true value must be $n+1$.

□

Theorem 3.3. *The chromatic number of a given web graph $w_{n,m}$ is 3 if m is even, and 4 if m is odd.*

Proof. Suppose $W_{n,m}$ is a particular but arbitrarily chosen web graph and suppose $c(W)$ is the chromatic number of $W_{n,m}$. By the definition of a chromatic number, $c(W)$ is the smallest number of colors needed to color the vertices of the graph so that no two adjacent vertices share the same color.

By the definition of a web graph, the edge set of $W_{n,m}$ is

$$E(W_{n,m}) = \bigcup_{i=1}^{a} E_i \cup \bigcup_{i=1}^{b} F_i \cup H.$$

In this set, the subset H represents the edges connecting v_0, the bullseye vertex, to the vertices in the 1st ring. By the definition of H,

$$H = \{\{v_0, v_i^1\} | 1 \leq i \leq m\}$$

Since all vertices on the ring are adjacent to v_0, the bullseye vertex must be a different color then the edges on the ring. This means $c(W)$ must be at least 2.

Next, E_i, as defined in our edge set, tells us by the definition of E_i,

$$E_i = \{\{v_k^i, v_{k+1}^i\} | 1 \leq k \leq m-1\} \cup \{v_1^i, v_m^i\}.$$

This shows that on each ring there is a circuit. If a given vertex on a ring is the "kth" vertex, then there is an adjacent $k-1$ vertex and an adjacent $k+1$ vertex. The "kth" vertex must be a different color then its adjacent $k-1$ and $k+1$ vertices. However, the $k-1$ vertex and $k+1$ are not adjacent so they can be the same color. On the 1st ring, all the vertices are adjacent to the bullseye, and adjacent vertices must have different colors. Therefore, $c(W)$ must be at least 3.

According to the definition of a web graph, on a given ring, there are m vertices. These vertices make up a circuit c. There are two cases shown below where the circuit c can have a differing chromatic number.

Case 1 (m is an even number): If m is an even number, c can alternate two colors amongst the vertices, and we will end with a different color than we started with. Referring back to E_i in the edge set of a web graph, this ensures that the vertices in the edge $\{v_1^i, v_m^i\}$ are different colors, where m is the number of the last vertex in c. Therefore, on a given ring, if m is even, only two colors are needed. This means $c(W)$ for a given ring is 2, and our $c(W)$ is still 3.

Case 2 (m is an odd number): If m is an odd number, on each ring or circuit c, if we alternate two colors amongst the vertices, and we will end with the same color than we started with. This will not work because it will make the adjacent vertices in the edge $\{v_1^i, v_m^i\}$ the same color. Therefore, a third color is needed for the last vertex in c. Since three colors are needed on each ring, $c(W)$ for the graph must be 4.

Lastly, we must show that the interring edges do not play a role in the chromatic number. As defined in the edge set, the interring edges are represented as F_i, where $F_i = \{\{v_i^k, v_i^{k+1}\} 1 \leq k \leq n\}$. F_i shows that the i number vertex on each ring k, is connected to the i number vertex on the $k+1$ ring. If $n > 1$, then we know that the vertices of ring 2 cannot have the same coloring pattern as the vertices on ring 1. This pattern will continue and we can say that the vertices of adjacent rings cannot be the same color. If we make the vertices of odd-numbered rings have the same colors and the vertices of even-numbered rings have the same colors, then we can ensure that the chromatic number will not increase because interring edges won't increase the chromatic number. If v represents the vertex, k is the number ring, and j its position on the ring, we can define the coloring of vertices on rings > 1 as:

Color v_j^k the same as v_j^1 if k is odd, or v_{j+1}^1 if k is even for $1 \leq j \leq m$.

This essentially colors all the vertices of all odd rings as the same as the innermost ring and colors the vertices of all even rings as the innermost ring, but starting with the $j + 1$ vertex, which has a different color. This means that adjacent vertices connected by interring edges will be different colors. Therefore, all possible vertices have been colored and $c(W)$ has not increased. This means that $c(W)$ is still 3 if m is even and 4 if m is odd. □

Theorem 3.4. *The independence number of a Web Graph $W_{n,m}$ is defined as*

$$\alpha(W_{n,m}) = n \cdot \left\lfloor \frac{m}{2} \right\rfloor$$

Proof. Let $W_{n,m}$ be a particular but arbitrary Web graph with n rings and m vertices on each ring.

Case 1: $n = 1$. When $n = 1$, we have a wheel graph W_{m+1} with $m + 1$ vertices. By definition, W_{m+1} = complete graph of 1 vertex, K_1, + C_m^1. By Theorem 2.6 in [1], $\alpha(W_{m+1}) = \left\lfloor \frac{m}{2} \right\rfloor$.

Case 2: $n > 1$. Let I be the maximal independence set of $W_{n,m}$. Let the center vertex $v_0 \notin I$, since if $v_0 \in I$, none of the m vertices in C^1 can be in I. Given *case* 1, we can select $\lfloor \frac{m}{2} \rfloor$ non-adjacent vertices in C^1 to be in I. It follows that there are $\lfloor \frac{m}{2} \rfloor$ vertices in C^2 that cannot be in I since they are adjacent to those selected in C^1 by the interring edges.

This means there are $m - \lfloor \frac{m}{2} \rfloor$ vertices in C^2 that may be selected. So, among those vertices, since $(m - \lfloor \frac{m}{2} \rfloor) \geq \lfloor \frac{m}{2} \rfloor$ we know that $\lfloor \frac{m}{2} \rfloor$ vertices that are non-adjacent and thus can be added to I. Proceeding in this way for every ring C^i_m, where $2 \leq i \leq n$, we have a total of n rings with each $\lfloor \frac{m}{2} \rfloor$ non-adjacent vertices. So the size of I is $|I| = \alpha(W_{n,m}) = n \cdot \left\lfloor \dfrac{m}{2} \right\rfloor$.

4. Conclusion

This paper successfully analyzed two distinct graph structures: a predefined Book graph and a newly defined Web graph. We explored the properties of these graphs including their order, size, minimum and maximum degree, chromatic number, dominating number, and independence number. Theorems were proven to solidify these properties for both the Book and Web graphs. These analyses contribute to the current understanding of graph theory by introducing a novel web graph structure and exploring its characteristics alongside a well-established Book graph.

□

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